Power- and Bandwidth-Efficient Communications Using LDPC Codes

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***Abstract*—We apply low-density parity-check (LDPC) codes to a bandwidth-efficient modulation scheme using multilevel coding, multistage decoding, and trellis-based signal shaping. Performance results based on density evolution and simulations are presented. Using irregular LDPC component codes of block length 10^5 and a 64-quadrature amplitude modulation signal constellation operating at 2 bits/dimension, a bit-error rate of 10^{-5} is achieved at an E_b/N_0 of 6.55 dB. At this value of E_b/N_0, the Shannon channel capacity, computed assuming equally likely signaling, is below 2 bits/dimension.**

***Index Terms*—Density evolution, low-density parity-check (LDPC) codes, multilevel coding (MLC), trellis shaping.**

I. Introduction

MULTILEVEL CODING (MLC) with multistage decoding (MSD) is a powerful coded modulation scheme capable of achieving power- and bandwidth-efficient communication by adapting channel coding to the transmission of an M-ary signal constellation [1]. It is known that MLC together with MSD can achieve the channel capacity, provided the component codes are chosen appropriately [2]. Since low-density parity-check (LDPC) codes [3]–[5] have been shown to have excellent performance on the additive white Gaussian noise (AWGN) channel, we consider using binary LDPC codes as component codes in an MLC scheme. The sum–product message-passing algorithm [4] is implemented in the MSD scheme to decode the LDPC component codes.

It is well known that when the signaling constellation is not symmetric, channel capacity cannot be achieved using each signal point with equal probability. In order to achieve a higher coding gain under these situations, we combined MLC/MSD with trellis shaping [6]. The density-evolution technique [4] is extended and used to design good irregular LDPC component codes suitable for the MLC/MSD/trellis-shaping system. By using nearly optimum code rates at each level, together with well-designed irregular LDPC codes, we show that the MLC/MSD/trellis-shaping system has excellent bit-error rate (BER) performance with reasonable computational complexity.

Several authors have studied the combination of LDPC codes and coded modulation. The recent paper by Narayanan and Li [7] shows the result of using short block length, regular LDPC codes with MLC/MSD and M-phase-shift keying (PSK) modulation. Due to the difficulty of constructing good LDPC codes of very high rate and small block lengths, a Bose–Chaudhuri–Hocquengem (BCH) code was used at the highest coding level of the MLC/MSD system. The performance obtained was within 1 dB of the channel capacity. The design of irregular LDPC codes for MLC with parallel independent decoding (PID) and M-pulse-amplitude modulation (PAM) has also been studied by Hou *et al.* [8]. The performance achieved was very close to the PID capacity for equiprobable 4-PAM, but remains far from the channel capacity, since the PID scheme together with equally probable signaling is suboptimum. For example, in Hou's paper, the optimized irregular LDPC codes with MLC/PID achieved a BER of 10^{-6} at a signal-to-noise ratio (SNR) lying about 0.13 dB away from the PID capacity with equally likely signaling. This capacity, in turn, is an additional 0.16 dB away from the AWGN channel capacity of a 4-PAM signal set when operating at a spectral efficiency of 1 b/symbol with equally likely signaling. Moreover, the irregular LDPC codes in this case had high degrees leading to high decoding complexity. Recently, Varnica *et al.* [9] implemented a concatenated coding scheme for the quadrature amplitude modulation (QAM) AWGN channel that achieves both shaping and coding gain. This concatenated coding scheme consisted of an inner trellis code and outer LDPC component codes in an MLC/MSD framework. The inner trellis code was constructed to achieve shaping. Iterative sum–product decoding over a joint factor graph of the inner trellis code, combined with the outer LDPC component codes, was proposed and evaluated. Using a 40-state inner trellis code and high-degree irregular LDPC outer component codes of block length 10^6, this concatenated coding scheme with 256-QAM achieved a BER of 10^{-6} at a SNR within 0.8 dB of the ultimate Shannon capacity limit when operating at a rate of 5 b/symbol. Unlike Varnica's scheme, our approach [10] is not based on concatenated coding. We accomplish shaping within the MLC/MSD framework. The component codes at the lower levels are LDPC codes, and a simple trellis-shaping code is used at the highest level.

This letter is organized as follows. In Section II, we discuss the combination of trellis shaping with a MLC/MSD scheme. The method for determining the noise threshold based on the density evolution technique is also described. In Section III, we present simulation results and threshold calculations for the MLC/MSD/trellis-shaping system with 64-QAM signal constellation. Finally, Section IV summarizes our main results.

II. MLC/MSD Combined With Trellis Shaping

A. System Model

In this letter, we consider nonuniform signaling of $M = 2^L$-ary QAM constellation using MLC/MSD with

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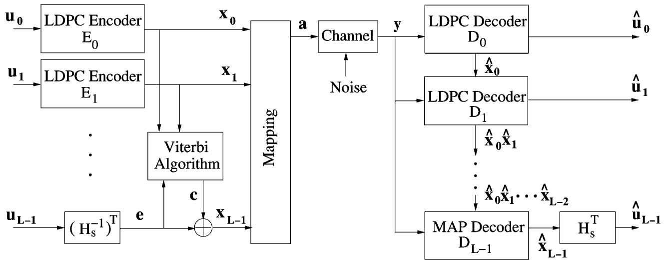

Fig. 1. Combination of MLC/MSD and trellis shaping.

L coding levels. In addition, nonuniform signaling is implemented by using a trellis-shaping scheme at the highest coding level, $L-1$. The principle of trellis shaping is to select the symbols of a QAM constellation with nonequal probability, i.e., the symbol points of lower energy are chosen with higher probabilities than those associated with higher energy signal points. Fig. 1 illustrates the combination of MLC/MSD and trellis shaping. The channel output y is given by $y = a + n$, where n is zero-mean Gaussian noise of standard deviation σ. Thus, this is a model of an AWGN channel with a matched-filter receiver and a one-dimensional signaling constellation.

First, we note that the lower $L-1$ levels are coded using LDPC codes of block length N, as in the case of no shaping. Shaping is only implemented at the highest level where no error-correcting code is applied. Below, we discuss the shaping operation.

A convolutional shaping code C_s can be specified by a $K_s \times N_s$ generator matrix G_s or by an $(N_s - K_s) \times N_s$ parity-check matrix H_s, where $G_s H_s^T = 0$ and the elements of these matrices are polynomials. Let $(H_s^{-1})^T$ denote an $(N_s - K_s) \times N_s$ left inverse of H_s^T, i.e., $(H_s^{-1})^T (H_s^T) = I$, where I is the $(N_s - K_s) \times (N_s - K_s)$ identity matrix. If $\mathbf{s} = \mathbf{e} H_s^T$ is the syndrome sequence associated with some error sequence $\mathbf{e}$, then $\mathbf{e} = \mathbf{s}(H_s^{-1})^T \oplus \mathbf{c}$ for some codeword $\mathbf{c} \in C_s$. Therefore, the syndrome $\mathbf{s}$ specifies one of the cosets, $C_s \oplus \mathbf{e}$, of the convolutional code. In the jth encoding time interval, $(H_s^{-1})^T$ is used to generate N_s bits $\mathbf{e}$ from $N_s - K_s$ information bits, $\mathbf{u}_{L-1}$, using $\mathbf{e} = \mathbf{u}_{L-1}(H_s^{-1})^T$. The bit sequence, $\mathbf{x}_{L-1}$, is given by $\mathbf{x}_{L-1} = \mathbf{e} \oplus \mathbf{c}$, where $\mathbf{c} \in C_s$. The choice of $\mathbf{c}$, given LDPC codewords at lower levels, $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{L-2}$, and the bit sequence $\mathbf{e}$, will determine the energy of the transmitted symbol sequence $\mathbf{a}$. The Viterbi algorithm is used to search the paths through the convolutional code trellis in order to find the path and the corresponding codeword $\mathbf{c}$, which minimizes the energy of the transmitted symbol sequence. This is accomplished by assigning to each branch in the convolutional code trellis a metric proportional to the energy of the corresponding transmitted symbol. The corresponding information bit sequence $\mathbf{u}_{L-1}$ can always be recovered from the bit sequence $\mathbf{x}_{L-1}$, since $\mathbf{u}_{L-1} = \mathbf{x}_{L-1} H_s^T$ for every choice of $\mathbf{c} \in C_s$. Thus, in addition to inducing a nonuniform probability distribution onto the signal constellation, the shaping code conveys information with a code rate of $R_s = 1 - K_s/N_s$ information bits per shaped bit.

At the receiver side, the LDPC decoder (sum–product decoder) at each level $i, i \in \{0, 1, \ldots, L-2\}$, uses the received signal y, the information provided by the lower levels to estimate the transmitted codeword at level $i, \mathbf{x}_i$, and the corresponding information sequence for that level, $\mathbf{u}_i$. At the highest level, $L-1$, a maximum *a posteriori* (MAP) estimate of the shaped bit, $\hat{x}_{L-1}$, is made by computing the bit w, which maximizes the probability $P(x_{L-1} = w \,|\, x_0 = \hat{x}_0, x_1 = \hat{x}_1, \ldots, x_{L-2} = \hat{x}_{L-2}, y)$. The estimated information bit sequence $\hat{\mathbf{u}}_{L-1}$ can now be reconstructed by performing the operation $\hat{\mathbf{u}}_{L-1} = \hat{\mathbf{x}}_{L-1} H_s^T$. Even if there are occasional errors in $\hat{\mathbf{x}}_{L-1}$, this will only cause limited error propagation in the estimated information sequence $\hat{\mathbf{u}}_{L-1}$, since H_s^T can always be chosen to be feedback free. Finally, it is important to note that since shaping induces a nonuniform distribution on the frequency of use of the signal points, prior probabilities for each signal point have to be taken into account in the sum–product decoding algorithm.

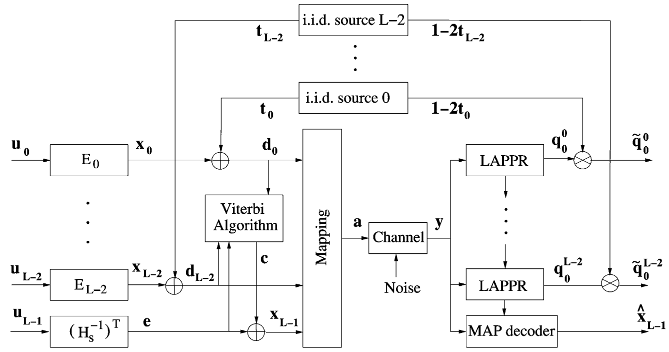

Fig. 2. MLC/MSD/trellis-shaping system with i.i.d. channel adapters on each of the equivalent binary-input channels.

B. Threshold Calculation

Due to the asymmetry of a QAM signal set, the ith, $i \in \{0, 1, \ldots, L-2\}$, binary-input component channel is not output symmetric [4], and thus, we cannot assume that the decoder errors are in the same positions, regardless of which codeword is transmitted. Therefore, when implementing the density evolution technique to determine the threshold, it is not valid to assume that the all-zero codeword is transmitted at each level. A recent paper by Hou *et al.* [8] developed a method to approximate the threshold of an LDPC code at each level used in an MLC/PID scheme with Gray mapping. The key idea was the introduction of "i.i.d. channel adapters," which symmetrize the equivalent binary-input component channels. We apply this method to the case of an MLC/MSD/trellis-shaping scheme with mapping by set partitioning [11].

Fig. 2 shows MLC/MSD combined with trellis shaping and i.i.d. channel adapters on each of the equivalent binary-input component channels. Each i.i.d. channel adapter has three parts: an i.i.d. source; a modulo-2 adder; and a multiplier. An i.i.d. source generates i.i.d. random variables $t_i \in \{0, 1\}, i \in \{0, 1, \ldots, L-2\}$ with $P(t_i = 0) = P(t_i = 1) = 1/2$. A modulo-2 adder adds the LDPC-coded bit x_i and the random number t_i to get $d_i = x_i \oplus t_i$. The multiplier performs the following operation: $\tilde{q}_0^i = q_0^i \cdot (1 - 2t_i)$, where q_0^i is the log *a posteriori* probability ratio (LAPPR) from the channel output at coding level i. The new equivalent binary-input component channel satisfies the required symmetry condition given by $p(\tilde{Q}_0^i = \tilde{q}_0^i \,|\, X_i = 0) = p(\tilde{Q}_0^i = -\tilde{q}_0^i \,|\, X_i = 1)$ as verified in [8], and hence, it can be assumed that the all-zero codeword is transmitted when evaluating system performance [4].

The initial message density at the ith coding level when using an i.i.d. channel adapter, i.e., the probability density function (pdf) of $\tilde{q}_0^i$, can be determined as follows. First, we note that the initial message, q_0^i, without using an i.i.d. channel adapter for coding level i is given by

$$q_0^i = \ln\left(\frac{\sum_{a\in\mathcal{I}} f(y\,|\,a)P(a)}{\sum_{a\in\mathcal{I}'} f(y\,|\,a)P(a)}\right) \tag{1}$$

where $f(y\,|\,a)$ denotes the channel transition pdf for receiving y given that signal a was transmitted, the set $\mathcal{I}$ is the set of signals at the partitioning level i that correspond to the coded bit 0, and the set $\mathcal{I}'$ denotes the set corresponding to the coded bit 1. After using the i.i.d. channel adapters, the initial message becomes $\tilde{q}_0^i = q_0^i$ if $t_i = 0$, and $\tilde{q}_0^i = -q_0^i$ if $t_i = 1$. Therefore, the initial message density is given by

$$p\left(\tilde{Q}_0^i = \tilde{q}_0^i\right) = \sum_{a\in\mathcal{I}} p\left(Q_0^i = \tilde{q}_0^i \,\middle|\, a\right) P(a) + \sum_{a\in\mathcal{I}'} p\left(Q_0^i = -\tilde{q}_0^i \,\middle|\, a\right) P(a). \tag{2}$$

In summary, we assume first that the signals from an M-ary amplitude-shift keying (ASK) signal set with the average signal energy E_s are transmitted over the AWGN channel using an MLC/MSD/trellis-shaping scheme. For each coding level $i, i \in \{0, 1, \ldots, L-2\}$, we fix the noise standard deviation at σ_i, corresponding to the SNR per symbol $E_s/N_0 = E_s/2\sigma_i^2$. The density evolution is then run with an input message distribution generated numerically, as described by (2). The algorithm is run iteratively until the error probability obtained from density evolution either approaches zero (practically reaches a very small value, we used 10^{-6}), or the number of iterations exceeds the preset value (e.g., 1000). The maximum value of the noise standard deviation at level i such that the error probability approaches zero is the noise threshold, σ_i^*. Since the last level of the MLC/MSD/trellis-shaping scheme contains no LDPC code, an upper bound on a BER at this level needs to be derived as a function of the noise standard deviation σ of the AWGN channel. From this bound, described in Section III, we can find the value of σ such that the BER at this last level is guaranteed to be very small, say 10^{-6}. This σ is then used as the threshold for the last level, and is denoted by σ_{L-1}^*. The threshold of the overall MLC/MSD/trellis-shaping system using LDPC codes and the sum–product decoding algorithm is given by

$$\sigma^* = \min_{0\le i\le L-1} \sigma_i^*. \tag{3}$$

This condition guarantees that at the noise standard deviation σ^*, the probability of decoding error for each level is very small. As a result, the probability of decoding error for the MLC/MSD/trellis-shaping system is very small at this noise standard deviation, as well.

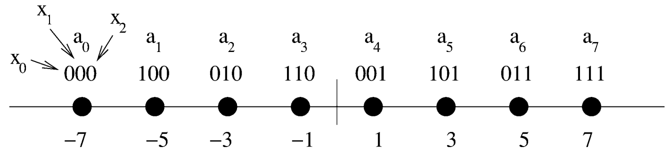

Fig. 3. Mapping by set partitioning of an 8-ASK signal set, assuming $\mathbf{A} = \{\pm 1, \pm 3, \pm 5, \pm 7\}$.

III. Simulation Results

In order to determine the optimal rate assignment for each LDPC component code based on the equivalent channel capacity rule, the probabilities associated with each signal point need to be known [2]. Even though these probabilities can be determined by using the Blahut–Arimoto algorithm [12], [13], this method involves extensive calculations. Hence, for simplicity, we have chosen the probability distribution $P(a)$ to be Gaussian, which is nearly optimum. Thus

$$P(a) = K(\gamma)e^{-\gamma|a|^2} \tag{4}$$

where $K(\gamma)$ is the normalizing factor, and the choice of $\gamma \ge 0$ provides a tradeoff between the average power of the signal set and its entropy, $H(A)$. The optimum γ, called γ^*, is the one that minimizes the gap between the Shannon capacity limit and the E_b/N_0 required to achieve reliable communication with input signals distributed with this Gaussian distribution. With this value of γ^*, the capacities of the equivalent channels can be determined. The assignment of rates for each level, therefore, follows directly from the equivalent channel capacity rule given in [2].

Simulations were performed for an MLC 8-ary ASK constellation combined with trellis shaping when used over the AWGN channel. The labeling of the 8-ASK constellation was based on Ungerboeck's partition rule (mapping by set partitioning) and is shown in Fig. 3. As indicated in [2], in order to approach capacity when shaping is implemented, nominally 1 bit of redundancy per dimension is required. Thus, we consider a total rate $R = 2.0$ b/dimension (dim) for this 8-ary ASK signaling constellation. At the rate of 2.0 b/dim, we find that the value $\gamma^* = 0.05$ minimizes the gap from capacity. The corresponding entropy of the signal set is $H(A) = 2.63$. Using this γ^* in (4), we get the probability of each signal point, which can then be used to calculate the equivalent channel capacities. The optimum rates in bits/channel use for each level are found to be $R^0 = 0.38, R^1 = 0.96$, and $R^2 = 0.66$. Note that, in general, for MLC/MSD without shaping, $R^0 \le R^1 \le R^2$. With shaping, R^2 is reduced to accommodate the shaping operation. LDPC codes are used for the first two levels, and the rate for the last level suggests the use of a shaping convolutional code of rate $1/3$. Therefore, $R_s = 1 - 1/3 = 2/3$. In all of our simulations, we have used a convolutional code of rate $1/3$ and constraint length 5 with the generator matrix $G_s = (25, 33, 37)_8$.

Fig. 4 shows our simulation results obtained for MLC/MSD with LDPC component codes for a 64-QAM constellation, based on two 8-ASK component constellations. The dashed lines correspond to the result of using LDPC codes of block length 10^4, whereas the solid lines correspond to codes of block length 10^5. The right-most dashed curve is the simulation

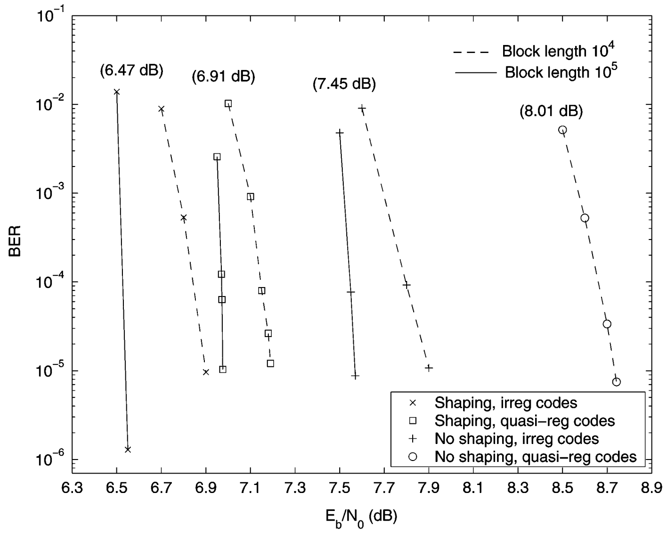

Fig. 4. Simulation results of MLC/MSD with LDPC component codes. A 64-QAM constellation based on two 8-ASK constellations has been used at a rate of 2 b/dim.

result obtained using quasi-regular LDPC codes[1] without trellis shaping. There is no shaping in this case, and the optimized rates given by the equivalent channel capacity rule are $R^0 = 0.18, R^1 = 0.82$, and $R^2 = 1.00$. We used a rate-0.18 $(3, 3\,|\,4)$ and a rate-0.82 $(3, 16\,|\,17)$ quasi-regular LDPC codes as component codes at levels 0 and 1, respectively. By using the differential evolution technique [14], we designed an irregular LDPC code[2] of rate 0.18 with a maximum bit node degree of 7 for coding level 0. The polynomials $\lambda(x)$ and $\rho(x)$ for this code are $\lambda(x) = 0.385\,107x + 0.127\,537x^2 + 0.487\,356x^6$ and $\rho(x) = 0.996\,888x^3 + 0.003\,112x^4$, respectively. The figure indicates that for codes of block length 10^4, by switching from a quasi-regular to an irregular LDPC code, an additional coding gain of 0.83 dB can be achieved at a BER of 10^{-5}. Furthermore, an additional coding gain of 0.34 dB can be realized by increasing the block length of irregular codes to 10^5.

In order to achieve better performance, trellis shaping is combined with MLC/MSD. As depicted in Fig. 4, an additional coding gain of 0.70 dB can be attained at a BER of 10^{-5} for codes of block length 10^4, by using quasi-regular codes with the appropriate rates, over the case where no shaping and irregular codes are used. We used a rate-0.38 $(3, 4\,|\,5)$ and a rate-0.96 $(3, 64\,|\,65)$ quasi-regular LDPC codes as the component codes at levels 0 and 1, respectively. By designing a good irregular LDPC code for level 0 suitable for the MLC/MSD/trellis-shaping system, an additional coding gain of 0.3 dB is possible for a code of block length 10^4. This irregular code has a maximum bit node degree of 7 and a rate of 0.38, with $\lambda(x) = 0.328\,354x + 0.286\,541x^2 + 0.385\,105x^6$ and $\rho(x) = 0.853\,527x^4 + 0.146\,473x^5$. In fact, for an irregular code of block length 10^5, a BER of 10^{-5} is achieved at an E_b/N_0 of 6.55 dB. Note that the E_b/N_0 required to achieve reliable communication (according to channel capacity) across this channel, with equal probable signaling at a rate of 2 b/dim, is larger, i.e., $E_b/N_0 = 6.60$ dB. With nonequiprobable signaling, the Shannon channel capacity at 2 b/dim is 5.74 dB. By increasing the maximum bit node degree of the irregular LDPC component codes, we would expect to achieve performance closer to this ultimate limit.

In these simulations for codes of block length 10^4, we constructed the generator matrix and encoded the randomly generated information bits. The BER was then determined by decoding. For codes of block length 10^5, however, it was very difficult to construct the generator matrix from the parity-check matrix. Consequently, we used the MLC/MSD/trellis-shaping model with i.i.d. channel adapters, as shown in Fig. 2, and transmitted the all-zero codeword. As a result, the simulation results shown in Fig. 4 for block sizes of 10^5 are actually the fraction of codewords which are decoded in error, rather than the information BER, which will be less. Based on our observations during the simulations, we believe that the results of these two schemes, with and without i.i.d. channel adapters, are similar.

The numbers that appear in the parentheses above each curve in Fig. 4 are the threshold values for the corresponding coding schemes determined from the method described earlier. An upper bound on the BER at the last level of the MLC/MSD/trellis-shaping system can be determined from

$$\mathrm{BER} = \sum_{m=0}^{M-1} P_{L-1}(\mathrm{error}\,|\,a_m)P(a_m) \tag{5}$$

where $P_{L-1}(\mathrm{error}\,|\,a_m)$ is the bit-error probability at level $L-1$, given that the symbol a_m is transmitted. Since $P_{L-1}(\mathrm{error}\,|\,a_m), m \in \{0, 1, \ldots, M-1\}$ can be calculated similarly for each m, we only give the calculation of $P_{L-1}(\mathrm{error}\,|\,a_0)$ as an example. Based on the mapping of the 8-ASK signal set, we have

$$\begin{aligned} P_{L-1}(\mathrm{error}\,|\,a_0) &= P\left[\ln\left(\frac{f(y\,|\,a_4)P(a_4)}{f(y\,|\,a_0)P(a_0)}\right) > 0 \,\middle|\, a_0\right] \\ &= P\left[\frac{e^{-(y-1)^2/2\sigma^2}}{e^{-(y+7)^2/2\sigma^2}} > \frac{P(a_0)}{P(a_4)} \,\middle|\, a_0\right] \\ &= P\left[Y > \frac{2\sigma^2 \ln\left(P(a_0)/P(a_4)\right) - 48}{16} \,\middle|\, a_0\right]. \end{aligned}$$

Given that a_0 is transmitted, Y is a Gaussian distributed random variable with mean -7 and variance σ^2, according to the mapping in Fig. 3. Hence

$$P_{L-1}(\mathrm{error}\,|\,a_0) = Q\left(\frac{\frac{2\sigma^2\ln(P(a_0)/P(a_4))-48}{16} + 7}{\sigma}\right).$$

A numerical evaluation of (5) shows that an E_b/N_0 of less than 5.5 dB is required to achieve a BER of 10^{-6}. It turns out that this 5.5 dB is lower than the E_b/N_0 thresholds obtained using the density evolution for LDPC codes at levels 0 and 1. Therefore, the threshold of the overall MLC/MSD/trellis-shaping system is basically determined by the first two levels.

[1] A $(d_v, d_c - 1\,|\,d_c)$ quasi-regular LDPC code has a parity-check matrix with d_v ones in each column, and $d_c - 1$ or d_c ones in each row.

[2] An irregular LDPC code can be described by a bipartite graph of bit nodes and check nodes. The polynomials $\lambda(x)$ and $\rho(x)$ specify the edge distribution of the graph [5].

IV. Conclusion

In this letter, we have presented a communication system based on MLC/MSD combined with trellis shaping using binary LDPC component codes. The density evolution technique has been modified and adapted to analyze the performance of this system. Simulation results show that the AWGN channel capacity computed assuming equiprobable signaling can be achieved using nonequiprobable signaling with optimized irregular LDPC component codes of block length 10^5 in an MLC/MSD/trellis-shaping scheme. Furthermore, good performance using well-designed irregular LDPC component codes of much shorter block lengths has also been demonstrated. Although the delay caused by the multistage decoder structure can be a drawback, this coding system provides both power and bandwidth efficiency, as well as complexity advantages.

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